THE TOPOLOGIES ON THE FIELD OF INTEGERS MODULO P FOR WHICH MODULAR MULTIPLICATION IS CONTINUOUS

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It is the purpose of this thesis to examine the field of integers modulo p under the binary operations of modular addition and multiplication and to determine all topologies on this field for which modular multiplication is continuous.

It is well known [see J. Kelley, General Topology, D. Van Nostrand Co., Inc., New York, 1955, p. 106] that there are nontrivial topologies on the real numbers making this set a topological ring (e.g., the usual topology). The question arose as to whether or not there are finite topological rings. Since the set of real numbers is a field, it seemed natural to look at the finite field of integers modulo p. Although not proven in this thesis, the only topologies for which the set of integers modulo p is a topological ring seem to be the two trivial topologies; i.e., the indiscrete and the discrete topologies. Modular addition proved to be the problem in that it appeared to be continuous for only the two trivial topologies. However, modular multiplication was much more interesting in that there were many more topologies for which it was continuous. This seemed to be a logical direction to explore.

In Chapter II and Chapter III the preliminary definitions, theorems, and corollaries of abstract algebra and topology are presented. These results are needed to prove and understand the major theorems that appear in Chapter IV and Chapter V.

In Chapter IV the sufficient condition of the major theorem is established. More precisely, in Theorem I it is proven that modular multiplication is continuous on X, the field of integers modulo p, if $\mathcal{T}$ is one of the following topologies: (a) $\{X, \emptyset\}$; (b) $\{X, \emptyset, C\{0\}\}$; (c) $\{X, \emptyset, \{0\}\}$; (d) $\{X, \emptyset, C\{0\}, \{0\}\}$; (e) a basis for $\mathcal{T}$ is $\{X, \emptyset, C\{0\}\} \cup \{bA \mid b \in C\{0\}, A \text{ is a subgroup of } C\{0\}, A \neq C\{0\}\}$; (f) a basis for $\mathcal{T}$ is $\{X, \emptyset, \{0\}\} \cup \{\{0\} \cup bA \mid b \in C\{0\}, A \text{ is a subgroup of } C\{0\}, A \neq C\{0\}\}$; and (g) a basis for $\mathcal{T}$ is $\{X, \emptyset, \{0\}, C\{0\}\} \cup \{bA \mid b \in C\{0\}, A \text{ is a subgroup of } C\{0\}, A \neq C\{0\}\}$.

In Chapter V the necessary condition is presented. By the exhaustive proofs of Theorem II and Theorem III it is shown that if modular multiplication is continuous on the field of integers modulo p, then $\mathcal{T}$ is one of the topologies of Theorem I. Consequently, a necessary and sufficient condition for modular multiplication to be continuous is that $\mathcal{T}$ be one of the above-described topologies. Moreover, the number of different topologies for which modular multiplication is continuous over the field of integers modulo p is $4 + 3N$ where N is one less than the number of factors of $p - 1$.